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ELASTIC MATERIAL CONSTANTS FOR ISOTROPIC GRANULAR SOLIDS WITH PARTICLE ROTATION

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Abstract—The granular material perceived as a collection of particles is modelled as a micropolar continua taking account of particle interaction and microstructure of the material. Explicit expressions of constitutive constants in terms of elastic inter-particle stiffness are derived for granular solids with isotropic random fabric distribution. Based on the derived expressions for isotropic material, six material constants are identified instead of two constants used in conventional elasticity. The derived constitutive constants in explicit terms of inter-particle contact properties provide a fundamental understanding of these constants and furnish useful inter-relations among these constants. The physical meaning of these material constants is discussed with emphasis on the role of particle spin. Finite element analysis incorporating these material constants is described and used to obtain solutions for boundary value problems. Examples are given for granular solids under boundary pressure to show the effects of inter-particle properties and material constants on the behavior of stress distribution and deformation in granular material.

1. INTRODUCTION

In earlier attempts, we have represented discrete granular material as equivalent continua by treating particle translation and particle rotation as two continuum fields (Chang, 1989; Chang and Liao, 1990) and subsequently derived constitutive relations considering the effect of particle interactions and microstructural properties (Chang and Ma, 1991). The derivation leads to a constitutive relationship which resembles micropolar type (Toupin, 1964; Green, 1965; Eringen, 1968). The micromechanical based approach is novel in that the derived constitutive constants are explicitly in terms of inter-particle properties.

In this paper we aim to derive close-form expressions for the macroscopic constitutive constant in terms of microstructural properties. Close-form expressions are desirable because they provide a better understanding of the influence of inter-particle properties on the stress–strain relationship. However, close-form expressions can only be obtained for simplified conditions since it involves complicated mathematical operations. Thus in this paper we deal with a random packing of equal-sized spheres with isotropic fabric distribution and linear elastic contact interaction between particles.

Expressions of constitutive constants thus derived consist of six material constants for isotropic granular solids. Of the six constants, two are identical to the Lamé constants of the elasticity. The other four represent additional material constants for granular material. The physical meaning of these constants will be evaluated to gain some insight into the mechanism of particle spin and to assess the influence of inter-particle properties.

To show the effects of these material constants and the inter-particle properties on the deformation behavior of granular solids in general boundary value problems, a finite element formulation is then described which incorporates the six constitutive constants. Using the finite element method, the behavior of deformation and stress distribution is discussed for granular solids under boundary pressure.

2. MODELLING OF GRANULAR MATERIAL AS EQUIVALENT CONTINUUM

2.1. *Mechanics of interactions between particles*

A simple conceptual model for granular material is to treat it as a collection of particles which are connected at contact points by imaginary springs (Duffy and Mindlin, 1957;

Walton, 1987). The spring in general is elasto-plastic. The elastic portion of the spring deformation is contributed by particle distortion while the plastic portion is contributed by the sliding between particles. For simplicity, in this paper, we confine our discussion to the elastic case only.

To represent the contact resistance, two types of springs are used, namely, the rotation springs and the stretch springs. The rotation springs, transmitting contact moments, represent the rolling and torsional resistance at inter-particle contacts. The stretch springs, transmitting contact forces, represent the compression and sliding resistance at inter-particle contacts.

The interaction between the nth particle and its surrounding particles is shown schematically in Fig. 1. When the assembly is subjected to an increment of load, particles undergo translations u_i and rotations ω_i, resulting in the stretch of springs at contacts. For example, at the cth contact of the nth particle as shown in Fig. 1, the angular rotation, θ_i^{cn}, and the stretch, δ_i^{cn}, of the spring are caused by the movement of the particle "c" relative to the particle "n", given by

$$\theta_i^{cn} = \omega_i^c - \omega_i^n \tag{1}$$

$$\delta_i^{cn} = (u_i^c - u_i^n) + \Xi_{ijk}(\omega_j^c r_k^{cn} - \omega_j^n r_k^{nc}) \tag{2}$$

where r_k^{nc} and r_k^{cn} are vectors joining the centroids of particles "n" and "c" respectively to the contact point between the two particles as shown in Fig. 1. The quantity $\Xi_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$ is the permutation of symbols used in tensor representation for the cross product of vectors.

Due to the stretch and rotation of the springs at a contact, corresponding moment, $\mathrm{d}m_i^c$, and force, $\mathrm{d}f_i^c$, are developed, given by

$$\mathrm{d}m_i^c = g_{ij}^c\,\mathrm{d}\theta_j^c \tag{3}$$

$$\mathrm{d}f_i^c = k_{ij}^c\,\mathrm{d}\delta_j^c \tag{4}$$

where g_{ij}^c is the stiffness of the rotation springs and k_{ij}^c is the stiffness of the stretch springs.

If the stiffness tensors g_{ij}^c and k_{ij}^c are decoupled, they take the form,

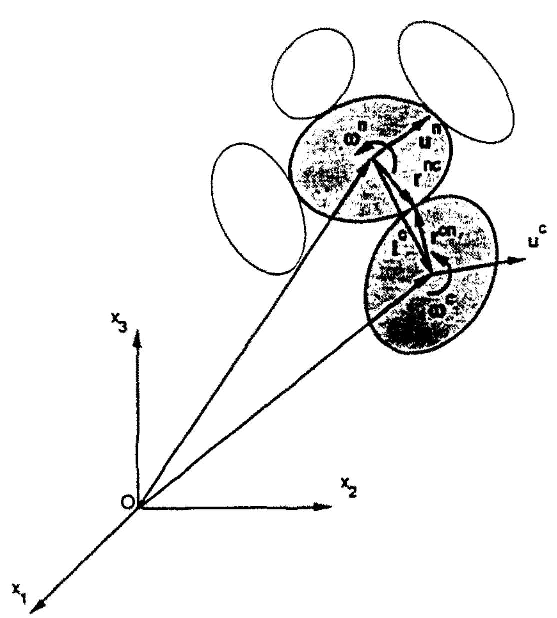

Fig. 1. Schematic figure for particle interaction.

$$q_{ij}^c = g_n^c n_i^c n_j^c + g_s^c s_i^c s_j^c + g_t^c t_i^c t_j^c \quad (5)$$

$$k_{ij}^c = k_n^c n_i^c n_j^c + k_s^c s_i^c s_j^c + k_t^c t_i^c t_j^c \quad (6)$$

where g_n^c and k_n^c, g_s^c and k_s^c, and g_t^c and k_t^c are the stiffness constants in the directions of the local coordinates n, s and t respectively. The local coordinate system is constructed for each contact with three orthogonal base unit vectors: the vector is normal and the vectors s and t are tangential to the contact area.

2.2. *Strain and stress in a granular solid*

In the macrocontinuum fields, in the neighborhood of the nth particle (i.e. the cluster shown in Fig. 1), we assume the usual "affine" (or homogeneous) deformation (Eringen, 1968). In Fig. 1, the nth particle is associated with four kinematic variables: u_i^n and ω_i^n as the displacement and rotation, and $u_{i,j}^n$ and $\omega_{i,j}^n$ as the derivatives of the displacement and the rotation, respectively.

For convenience, we introduce the asymmetric deformation strain ε_{ji} and the polar strain γ_{ji}^n in terms of variables related to particle displacement and rotation (Chang and Liao, 1990)

$$\varepsilon_{ji}^n = u_{i,j}^n - \Xi_{jik}\,\omega_k^n \quad (7)$$

$$\gamma_{ji}^n = \omega_{i,j}^n. \quad (8)$$

With the affine deformation assumption, the angular rotation, θ_i^{nc}, and the stretch, δ_i^{nc}, of the springs at the contact between particle "n" and particle "c" [eqns (1) and (2)] can be expressed by the strains defined for the particle [eqns (5) and (6)], as follows:

$$\theta_i^{nc} = \gamma_{ji}^n l_j^{nc} \quad (9)$$

$$\delta_i^{nc} = \varepsilon_{ji}^n l_j^{nc} + \Xi_{ijk}\gamma_{lj} l_l^{nc} r_k^{nc} \quad (10)$$

where the branch vector $l_j^{nc} = r_j^{cn} - r_j^{nc}$.

Corresponding to the strain defined in eqns (7) and (8), the stresses in the granular assembly can be defined using the principle of energy equivalence by equating the work done due to the deformation of contact springs and the work done in terms of stress and strain (Chang and Ma, 1991). Thus we obtain the Cauchy stress tensor σ_{ij}^n in terms of contact forces as

$$\sigma_{ij}^n = \frac{1}{2v^n}\sum_{nc} l_i^{nc} f_j^{nc} \quad (11)$$

and the couple stress tensor μ_{ij}^n in terms of contact forces and moments as

$$\mu_{ij}^n = \frac{1}{2v^n}\sum_{nc} l_i^{nc}(m_j^{nc} + \Xi_{jkl} r_k^{nc} f_l^{nc}) \quad (12)$$

where v^n is the volume of the microcell which includes the solid volume of the nth particle and its associated voids.

Next, we envision a representative volume of granular material which is sufficiently small when compared with the scale of the boundary value problem, yet consisting of a sufficiently large number of particles to be statistically representative of the behavior of the material.

The stress for such a representative volume can be defined based on the volume average of stresses at particle level as

$$\sigma_{ij} = \frac{1}{V}\sum_{a=1}^{N} v^n \sigma_{ij}^n = \frac{1}{2V}\sum_{a=1}^{N}\sum_{c} l_i^{nc} f_j^{nc} \tag{13}$$

where the volume of the assembly

$$V = \sum_{n=1}^{N} v^n.$$

In the double summation of eqn (12), each contact is counted twice. Note that the vector $l_i^c = l_i^{nc} = -l_i^{cn}$ and $f_i^c = f_i^{nc} = -f_i^{cn}$. The stress can be further expressed by the summation over all particle contacts in the representative volume. Thus

$$\sigma_{ij} = \frac{1}{V}\sum_{c}^{M} l_i^c f_j^c \tag{14}$$

where M is the total number of contacts and $\sum_c^M$ denotes a summation over all contacts. Similarly, let the vector $\zeta_i^{nc} = r_i^{cn} + r_i^{nc}$ and note that $\zeta_i^c = \zeta_i^{nc} = \zeta_i^{cn}$ and $m_i^c = m_i^{nc} = -m_i^{cn}$, the average couple stress

$$\mu_{ij} = \frac{1}{V}\sum_{c}^{M} l_i^c m_j^c + \frac{1}{2V}\sum_{c}^{M} l_i^c \Xi_{jkl}\zeta_k^c f_l^c. \tag{15}$$

It is noted that, for packings with equal size spheres, ζ_k^c is zero and the couple stress in eqn (15) is contributed solely through the moments at contacts.

2.3. *Constitutive law of granular medium*

Let ε_{kl} and γ_{kl} be the strains defined for the representative volume. Observed from the results of computer simulation, the assumption of uniform strain seems reasonable for packings of equal sized particles with the linear elastic contact property (Chang and Misra, 1990). With the assumption of uniform strain in the representative volume, the stress–strain relationship for the representative volume can be derived based on eqns (14), (15), (3), (4), (9) and (10) as

$$\begin{Bmatrix} \sigma_{ij} \\ \mu_{ij} \end{Bmatrix} = \begin{bmatrix} a_{ijkl} & b_{ijkl} \\ b_{klij} & c_{ijkl} \end{bmatrix} \begin{Bmatrix} \varepsilon_{kl} \\ \gamma_{kl} \end{Bmatrix} \tag{16}$$

where a_{ijkl}, b_{ijkl} and c_{ijkl} are the constitutive coefficients for the representative volume, given by

$$a_{ijkl} = \frac{1}{V}\sum_{c}^{M} l_i^c K^{c_{jl}} l_k^c \tag{17}$$

$$b_{ijkl} = \frac{1}{V}\sum_{c}^{M} \Xi_{lmn} l_i^c K^{c_{jn}} l_k^c r_m^c \tag{18}$$

$$c_{ijkl} = \frac{1}{V}\sum_{c}^{M} l_i^c l^{c_k} (G_{jl}^c + \Xi_{jnq}\Xi_{lmp} K_{nm}^c \zeta_p^c r_q^c) \tag{19}$$

where M is the total number of contacts in the representative volume.

3. CONSTITUTIVE CONSTANTS FOR ISOTROPIC RANDOM PACKINGS

Since the representative volume consists of a large number of particles, the constitutive constants [eqns (17)–(19)] can be expressed in integral form by introducing a density function. For packings of equal size spheres with isotropic directional distribution of inter-particle contacts, the density function in a spherical coordinate (Fig. 2) is $1/4\pi$ and it satisfies

$$\int_0^{\pi}\int_0^{2\pi} \frac{1}{4\pi} \sin\gamma \, \mathrm{d}\gamma \, \mathrm{d}\beta = 1. \tag{20}$$

Let M be the total number of contacts in the volume V, the number of contacts in the interval from γ to $\gamma+\mathrm{d}\gamma$ and β to $\beta+\mathrm{d}\beta$ is given by $M(1/4\pi)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta$. A summation of any function $F^c(\gamma,\beta)$ over all contacts can thus be expressed in an integral form as follows:

$$\sum_{c=1}^{N} Fc(\gamma,\beta) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} F^c(\gamma,\beta) M \sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta. \tag{21}$$

For the packing of equal spheres of radius r, the branch vector $l_i^c = 2rn_i^c$, where the unit contact normal vector n_i^c is a function of γ and β. Replacing the form of summation in eqns (17)–(19) by integral form, we obtain

$$a_{ijkl} = \frac{r^2 N}{\pi V}\int_0^{\pi}\int_0^{2\pi} A_{ijkl}^c(\gamma,\beta) M \sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta \tag{22}$$

$$b_{ijkl} = \frac{r^3 N}{\pi V}\int_0^{\pi}\int_0^{2\pi} B_{ijkl}^c(\gamma,\beta) M \sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta \tag{23}$$

$$c_{ijkl} = \frac{r^2 N}{\pi V}\int_0^{\pi}\int_0^{2\pi} C_{ijkl}^c(\gamma,\beta) M \sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta \tag{24}$$

where

$$A_{ijkl}^c(\gamma,\beta) = n_i^c(\gamma,\beta) K_{jl} n_k^c(\gamma,\beta) \tag{25}$$

$$B_{ijkl}^c(\gamma,\beta) = \Xi_{lmn} n_i^c(\gamma,\beta) K_{jn} n_k^c(\gamma,\beta) n_m^c(\delta\gamma,\beta) \tag{26}$$

$$C_{ijkl}^c(\gamma,\beta) = n_i^c(\gamma,\beta) G_{jl} n_k^c(\gamma,\beta). \tag{27}$$

By observing eqns (25) and (27), we can identify the following properties:

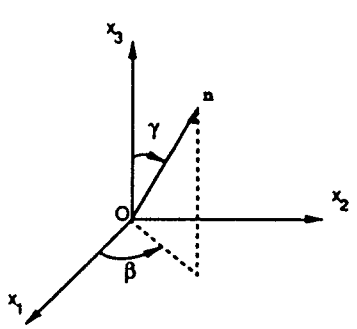

Fig. 2. Spherical coordinate for directional distribution of inter-particle contact.

$$a_{ijkl} = a_{klij} \quad c_{ijkl} = c_{klij}. \tag{28}$$

It is also noted that, for assembly with centro-symmetric packing, $b_{ijkl} = b_{klij} = 0$. Since granular material is statistically centro-symmetric, the stress–strain relationship given in eqn (16) decouples and yields to the following form:

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl} \tag{29}$$

$$\mu_{ij} = c_{ijkl}\gamma_{kl}. \tag{30}$$

3.1. *Stretch stress–strain relationship*

Close-form expressions of the stiffness constants a_{ijkl} for a random assembly of equal spheres with isotropic packing structure are obtained from eqn (22) assuming $k_s = k_t$. After mathematical operation, the constitutive constants can be expressed in the following form:

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + (G+z)\delta_{ik}\delta_{jl} + (G-z)\delta_{il}\delta_{jk}. \tag{31}$$

Or expressed in a matrix form as follows:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yx} \\ \sigma_{yz} \\ \sigma_{zy} \\ \sigma_{xz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda+2G & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G+z & G-z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G-z & G+z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G+z & G-z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G-z & G+z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G+z & G-z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G-z & G+z \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \\ \varepsilon_{yz} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \\ \varepsilon_{zx} \end{bmatrix}. \tag{32}$$

Equation (32) represents a material symmetry of isotropy where the three macro-material constants for a representative volume are expressed by micro-material constants k_n and k_t as follows:

$$G = 2\alpha(2k_n + 3k_t) \tag{33}$$

$$\lambda = 4\alpha(k_n - k_t) \tag{34}$$

$$z = 10\alpha k_t \tag{35}$$

where

$$\alpha = \frac{Mr^2}{30V}. \tag{36}$$

The value of α relates to both particle size and the number of contacts per volume M/V which is a function of void ratio, coordination number and particle size (Chang *et al.*, 1989). It is interesting to note that when $k_n < k_t$, λ becomes negative, leading to the behavior of a material with negative Poisson ratio. The material constant z in eqn (36) is not an independent constant. Observed from eqns (33)–(35), the constant

$$z = G - \lambda. \tag{37}$$

The strain and stress in general are asymmetric. The strain defined previously in eqn (7) is divided into two components: the symmetric part and the skew-symmetric part. The symmetric part,

$$\varepsilon_{(ij)} = \tfrac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji}) = \tfrac{1}{2}(u_{j,i} + u_{i,j}) \tag{38}$$

represents the stretch strain.

The skew-symmetric part of the strain

$$\varepsilon_{[ij]} = \tfrac{1}{2}(\varepsilon_{ij} - \varepsilon_{ji}) = \tfrac{1}{2}(u_{j,i} - u_{i,j}) - \Xi_{ijk}\omega_k \tag{39}$$

contains two terms: the skew-symmetric tensor $\Xi_{ijk}\omega_k$ representing the average rotation of particles in the representative volume and the skew-symmetric tensor $\tfrac{1}{2}(u_{j,i} - u_{i,j})$ representing rigid body rotation of the representative volume. The angular rotation ψ_k corresponding to the rigid body rotation is

$$\Xi_{ijk}\psi_k = u_{[j,i]}. \tag{40}$$

Substituting this equation into eqn (39), the skew-symmetric tensor

$$\varepsilon_{[ij]} = \Xi_{ijk}(\psi_k - \omega_k) \tag{41}$$

represents the value of the net average particle spin (i.e. the difference between particle rotation and the rigid body rotation of the assembly).

Similarly, we define the symmetric stress

$$\sigma_{(ij)} = \tfrac{1}{2}(\sigma_{ij} + \sigma_{ji}) \tag{42}$$

to represent the stretch and define the skew-symmetric stress

$$\sigma_{[ij]} = \tfrac{1}{2}(\sigma_{ij} - \sigma_{ji}) \tag{43}$$

to represent the momentum due to shear in the material.

Note that $\varepsilon_{[ij]} = -\varepsilon_{[ji]}$ and $\varepsilon_{(ij)} = \varepsilon_{(ji)}$, eqn (32) can be rearranged and expressed in a matrix form as follows:

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{(xy)} \\ \sigma_{(yz)} \\ \sigma_{(zx)} \\ \sigma_{[xy]} \\ \sigma_{[yz]} \\ \sigma_{[zx]} \end{bmatrix} = \left[\begin{array}{cccccc|ccc} \lambda+2G & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{array}\right] \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{(xy)} \\ \varepsilon_{(yz)} \\ \varepsilon_{(zx)} \\ \varepsilon_{[xy]} \\ \varepsilon_{[yz]} \\ \varepsilon_{[zx]} \end{bmatrix}. \tag{44}
$$

When the skew parts of stress and strain are neglected, the relationship between stress and strain [eqn (44)] reduces to a form identical to that in conventional theory of elasticity. The values of λ and G are thus the usual Lamé constants. Corresponding to the Lamé constants, the values of macro-material constants Young's modulus E and Poisson's ratio v can be obtained in terms of the micro-material constants k_n and k_t, given by

$$E = \frac{20\alpha k_n(2k_n+3k_t)}{4k_n+k_t} \tag{45}$$

$$v = \frac{k_n-k_t}{4k_n+k_t}. \tag{46}$$

For a possible range of k_t from 0 to infinity, the admissible range of Poisson's ratio is thus -1–0.25.

On the other hand, using eqns (44) and (45), the micro-material constants k_n and k_t can be expressed in terms of the macro-material constants E and v as follows:

$$k_n = \frac{E}{20\alpha(1-2v)} \tag{47}$$

$$k_t = \frac{E(1-4v)}{20\alpha(1+v)(1-2v)}. \tag{48}$$

These relationships are useful for back calculating the micro-material constants from the laboratory measurements of macro-behavior of material samples.

The skew part of strain in the present formulation is introduced by a mechanism of particle spin which is not considered in the conventional theory of elasticity. Non-symmetrical shear stress for the representative volume results from the spin of particles which activates shear forces transmitting through particle contacts. Therefore, z in eqn (44), termed as a "spin modulus", is directly related to tangential contact stiffness k_t as shown in eqn (35). The spin modulus can also be expressed in terms of Young's modulus and Poisson's ratio, given by

$$Z = \frac{(1-4v)E}{2(1+v)(1-2v)}. \tag{49}$$

Equation (49) shows that v must be less than 0.25 in order to keep z positive for isotropic packing. This is in agreement with the admissible range of v previously described in eqn (46). Thus the phenomenon of $v > 0.25$ is attributed to the anisotropy of samples.

When the couple stress is absent, σ_{ij} is symmetric (i.e. $\sigma_{[ij]} = 0$) which leads to the skew part of strain $\varepsilon_{[ij]} = 0$ according to eqn (44). Thus no spin is expected for isotropic granular material under symmetric stress conditions.

3.2. *Couple stress–strain relationship*

Next we discuss the constitutive constants c_{ijkl} in eqn (30) between couple stress and rotation gradient. Similar to the derivation of a_{ijkl}, this matrix also has the form of isotropic symmetry, given by

$$\begin{bmatrix} \mu_{xx} \\ \mu_{yy} \\ \mu_{zz} \\ \mu_{xy} \\ \mu_{yx} \\ \mu_{yz} \\ \mu_{zy} \\ \mu_{xz} \\ \mu_{zx} \end{bmatrix} = \begin{bmatrix} \lambda_r+2G_r & \lambda_r & \lambda_r & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_r & \lambda_r+2G_r & \lambda_r & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_r & \lambda_r & \lambda_r+2G_r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_r+z_r & G_r-z_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_r-z_r & G_r+z_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_r+z_r & G_r-z_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_r+z_r & G_r-z_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_r+z_r & G_r-z_r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_r+z_r & G_r-z_r \end{bmatrix} \times \begin{bmatrix} \gamma_{xx} \\ \gamma_{yy} \\ \gamma_{zz} \\ \gamma_{xy} \\ \gamma_{yx} \\ \gamma_{yz} \\ \gamma_{zy} \\ \gamma_{xz} \\ \gamma_{zx} \end{bmatrix}. \quad (50)$$

where

$$G_r = 2\alpha(2g_n+3g_t) \quad (51)$$

$$\lambda_r = 4\alpha(g_n-g_t) \quad (52)$$

$$z_r = 10\alpha g_t \quad (53)$$

where $\alpha = Mr^2/30V$.

According to eqn (50), when the rolling stiffness g_t and twisting stiffness g_n at inter-particle contacts are zero, no couple stress is permitted to transmit. Although values of g_t and g_n are negligible for a contact between two smooth spheres, the values can be considerably larger for spheres bonded at contacts by an agent such as cement. Therefore eqn (50) is more pertinent to bonded granular material.

Since the matrix form of c_{ijkl} in eqn (50) resembles the form of a_{ijkl} relating Cauchy stress and stretch strain, in analogy to the Young's modulus E and Poisson's ratio v, two material constants E_R and v_R can be similarly introduced for the relationships of couple stress and rotation gradient. Under the applied stress μ_{xx} while $\mu_{yy} = \mu_{zz} = 0$, the constants E_R and v_R are defined as follows:

$$E_R = \frac{\mu_{xx}}{\gamma_{xx}}; \quad (54)$$

$$v_R = \frac{\gamma_{xx}}{\gamma_{yy}}. \quad (55)$$

E_R represents the twisting resistance of a representative volume. v_R is the ratio of rotation gradients in the directions of two Cartesian axes.

Parallel to eqns (45) and (46), it yields the following representations:

$$E_R = \frac{20\alpha g_n g_n (3g_t + 2g_n)}{4g_n + g_t} \tag{56}$$

$$v_R = \frac{g_n - g_t}{4g_n + g_t}. \tag{57}$$

It is noted that, if the twisting stiffness constant g_n of the contact is zero, no couple stress can be transmitted through the motion of twisting between two spheres and it can be seen from eqn (56) that the torsional resistance of the representative volume $E_R = 0$.

The contact shear forces due to a rotation gradient γ_{xx} for particles rotating in the direction of the x-axis can transmit through rotational springs and cause twist motion of particles along the y and z-axes. This effect is reflected by the variable v_R. For $g_n = 0$, the value of $v_R = -1$.

It is certain that due to rotation gradients of particles in the representative volume, the ability to transmit couple stress lies in the inter-particle properties g_n, q_t.

3.3. *Two-dimensional condition*

Reducing to a two-dimensional condition, the system contains the following components of displacement and rotation:

$$u = (u_x, u_y, 0) \quad \text{and} \quad \omega = (0, 0, \omega_z).$$

For a plane strain condition, the following strains can be presumed to have vanished:

$$\varepsilon_{zz} = 0 \quad \text{and} \quad \varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0,$$

$$\gamma_{zz} = \gamma_{xx} = \gamma_{yy} = 0 \quad \text{and} \quad \gamma_{xy} = \gamma_{zx} = \gamma_{yx} = \gamma_{zy} = 0.$$

On the other hand, for a plane stress condition, the following stresses vanish:

$$\sigma_{zz} = 0 \quad \text{and} \quad \sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0,$$

$$\mu_{zz} = \mu_{xx} = \mu_{yy} = 0 \quad \text{and} \quad \mu_{xy} = \mu_{zx} = \mu_{yx} = \mu_{zy} = 0.$$

For example, the stress–strain relationship of the material for the plane strain condition is reduced to the following form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yx} \\ \mu_{xz} \\ \mu_{yz} \end{Bmatrix} = \begin{bmatrix} \lambda+2G & \lambda & 0 & 0 & 0 & 0 \\ \lambda & \lambda+2G & 0 & 0 & 0 & 0 \\ 0 & 0 & G+z & G-z & 0 & 0 \\ 0 & 0 & G-z & G+z & 0 & 0 \\ 0 & 0 & 0 & 0 & M_r & 0 \\ 0 & 0 & 0 & 0 & 0 & M_r \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{58}$$

where $M_r = G_r + z_r = 4\alpha(g_n + 4q_t)$.

4. FINITE ELEMENT FORMULATION

For two-dimensional plane strain conditions, the strain–displacement relation can be expressed in matrix form as follows:

$$\{\varepsilon\} = [\partial]\{u\} \tag{59}$$

where

$$\{\varepsilon\} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}, \varepsilon_{yx}, \gamma_{xz}, \gamma_{yz})^{\mathrm{T}}, \quad \{u\} = (u_x, u_y, \omega_z)^{\mathrm{T}}$$

and

$$[\partial] = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & \partial_x & -1 \\ \partial_y & 0 & 1 \\ 0 & 0 & \\ 0 & 0 & x \\ & & y \end{bmatrix}. \tag{60}$$

In an element which consists of several nodal points, the displacements of nodal points can be expressed by a nodal displacement vector $\{u^e\}$. The displacement field of the element can be constructed using a shape function $[\Psi(x, y)]$ such that the displacement at any point (x, y) within the element is given by

$$\{u(x, y)\} = [\Psi(x, y)]\{u^e\}. \tag{61}$$

The strain at point (x, y) can thus be related to the displacements of nodal points by the following expression:

$$\{\varepsilon(x, y)\} = [B]\{u^e\} \tag{62}$$

where $[B] = [\partial][\Psi]$.

For an element with volume Ω_e and surface Γ_e, the potential energy for the element is

$$\Pi = \tfrac{1}{2}\int_{\Omega_e} \{\varepsilon\}^{\mathrm{T}}\{\sigma\}\,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_e} \{u\}^{\mathrm{T}}\{t\}\,\mathrm{d}s \tag{63}$$

where $\{t\}$ is the traction on the surface Γ_e of the element.

Applying the principle of virtual work which requires the variation of the potential energy to be zero in order to preserve minimum potential energy on an element of the granular medium,

$$\begin{aligned} \delta\Pi = 0 &= \int_{\Omega_e} \delta\{\varepsilon\}^{\mathrm{T}}\{\sigma\}\,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_e} \delta\{u\}^{\mathrm{T}}\{t\}\,\mathrm{d}s \\ &= \int_{\Omega_e} \delta\{u^e\}^{\mathrm{T}}[B]^{\mathrm{T}}[C][B]\{u^e\}\,\mathrm{d}x\,\mathrm{d}y - \int_{\Gamma_e} \delta\{u^e\}^{\mathrm{T}}[\Psi]^{\mathrm{T}}\{t\}\,\mathrm{d}s. \end{aligned} \tag{64}$$

The above equation is satisfied for any variation $\delta\{u^e\}$. It leads to the following equation:

$$[K]\{u^e\} = \{F^e\} \tag{65}$$

where $[K]$ is the stiffness matrix of the element, given by

$$[K] = \int_{\Omega_e} [B]^T [C][B] \, dx \, dy \tag{66}$$

and the nodal forces $\{F^e\}$ due to applied traction $\{t\}$ are given by

$$\{F^e\} = \int_{\Gamma_e} [\Psi]^T \{t\} \, ds. \tag{67}$$

After the element stiffness matrix is established, boundary value problems can be solved using the usual finite element procedure.

5. EXAMPLES OF TWO-DIMENSIONAL GRANULAR MATERIAL PACKINGS

Granular solids under external load are analyzed using the finite element method presented here to investigate the effects of constitutive coefficients. The granular material is taken to be isotropic so that the constitutive matrix is of the form given in eqn (58).

The finite element mesh shown in Fig. 3 represents a pressure load on boundary surface of granular material. Because a symmetrical condition is assumed, Fig. 3 reveals only half of the picture on the right of the center line. The pressure is 1 lb in^{-2} (6.895 kPa) loaded on a length of 2 in (5.08 cm) on the surface. At the base, the displacements are specified to be zero in both directions. The displacements on the center-line and on the side-line are specified to be zero in the horizontal direction. All boundary points are free to rotate except for the points on the center-line. A plane strain condition is assumed.

Element stiffness matrix for each element is derived based on eqn (66) given in the previous section and then assembled into a global stiffness matrix. With the specified boundary condition, a set of simultaneous equations is formulated and solved for the displacements and rotations for all nodal points. This computation procedure is carried out for the example problem. Constitutive constants, E, v and z used for the packing is derived from the properties of inter-particle contacts.

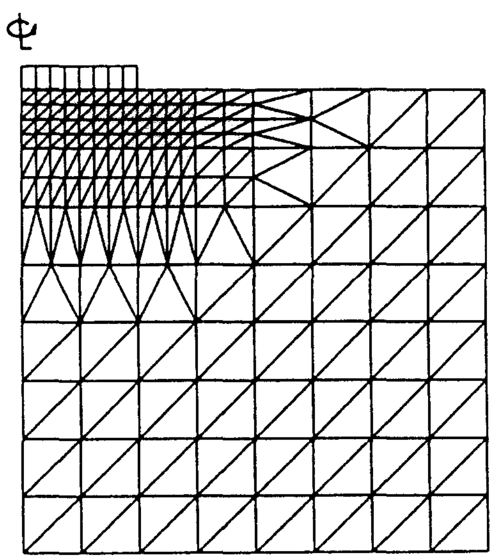

Fig. 3. Finite element mesh representing a pressure load on granular material.

Table 1. Material properties used in the example

Case	Inter-particle properties			Macro-properties		
	k_n^c—lb in^{-1}	k_t^c—lb in^{-1}	k_t^c/k_n^c	E—lb in^{-2}	ν	μ—lb in^{-2}
1	10	10	1	1000	0	500
2	12.5	6.9	0.55	1000	0.1	345
3	19.2	0.62	0.03	1000	0.24	31

$\alpha = 5$ for all cases; 1 lb in^{-1} = 0.18 kN m^{-1}; 1 lb in^{-2} = 6.895 kPa.

We first examine the cases with $M_r = 0$ (i.e. with no couple stress). Three combinations of k_n and k_t are used in the example. Their value and the corresponding moduli E, μ and Poisson's ratio ν are given in Table 1.

For the three cases, the computed deflections and stress 5% distribution deviate from that obtained in elasticity. According to the stress–strain relationship [eqn (44)], the present model under symmetric stress condition forces the rigid body rotation to be equal to the particle rotation (i.e. $\varepsilon_{[ij]} = 0$). This constraint, not imposed in the conventional elasticity theory, causes the discrepancies in the computed deflections and stresses.

Comparison of the vertical stress along the center-line versus depth, plotted for case 2, is shown in the upper part of Fig. 4 where $2B$ is the width of pressure load. The vertical stress versus horizontal distance at depth $2B$ is plotted in the lower part of Fig. 4. In the present model, the stress transmits to deeper depth and diminishes quicker in the horizontal direction. The settlement at the center of the pressure load computed by the present model are approximately 10% larger than that obtained from classical elasticity.

Next we examine the cases with various values of M_r to evaluate their effects on the deformation behavior. We solve the example problem using the same properties given in Table 1, while rotational stiffness of inter-particle contact g_n^c and g_t^c are varied to give various values of modulus M_r in eqn (58).

The settlement computed at the center of pressure load for the cases of $M_r = 0$ and $M_r \neq 0$ are denoted as δ_0 and δ_m respectively. Settlement ratio δ_m/δ_0 is plotted in Fig. 5 against a normalized factor m defined as follows:

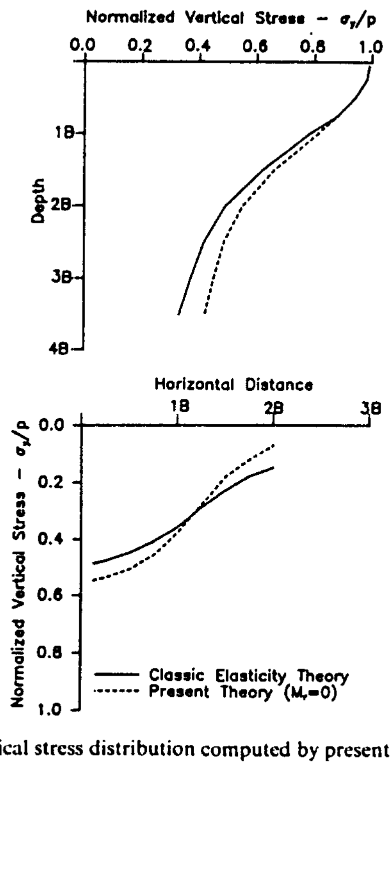

Fig. 4. Comparison of vertical stress distribution computed by present theory and elasticity theory.

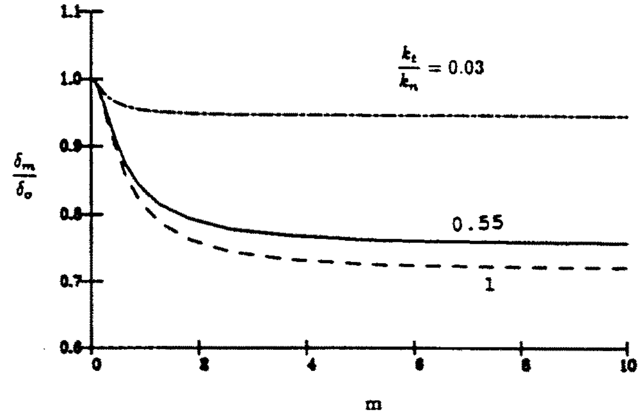

Fig. 5. Effects of rotational modulus and inter-particle stiffness on settlement.

$$m = \frac{1}{2B}\frac{M_r}{G}. \tag{68}$$

As shown in Fig. 5, it is expected that the higher stiffness for rotational springs (i.e. higher m) restrains the freedom of particle movement thus leading to a smaller settlement. Since couple stress generates asymmetric shear and particle spin which in turn affects the

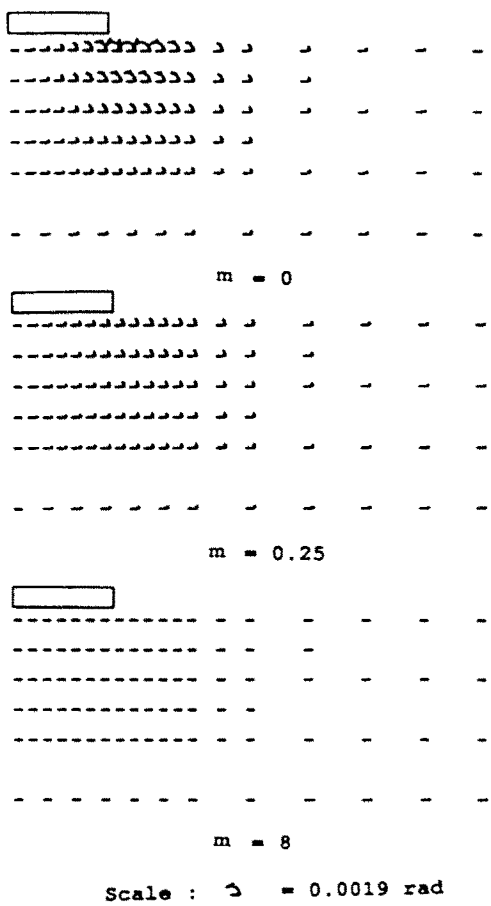

Fig. 6. Computed fields of particle rotation.

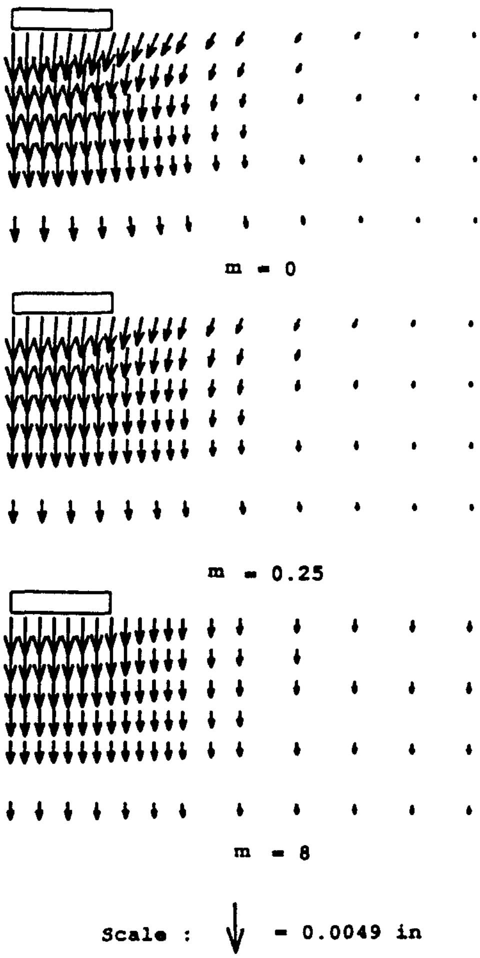

Fig. 7. Computed fields of particle displacement.

deformation of the material, the shear contact stiffness, k_t, is thus an important influencing factor. When $k_t/k_n = 1$, the settlement significantly decreases as m increases. When $k_t = 0$, the spin modulus $z = 0$ and the rotational modulus M_r has a smaller effect on the settlement. Most of the effect takes place when m changes from 0 to 2. As m is greater than 2, the rate of effect diminishes. The settlement continues to decrease slightly as m increases from 2 to 10.

For further discussion, we compare the example problem for the following three cases: (1) $m = 0$, (2) moderate modulus $m = 0.25$ and (3) high modulus $m = 8$. Computed rotations for nodal points are plotted in Fig. 6 for the three cases. For the case of $m = 0$, particle rotations are restrained merely by the contact tangential stiffness, while for higher rotation modulus particle rotations are subjected to additional constraint from rotational springs.

Differences in the computed displacement fields for the three conditions are displayed in Fig. 7. For $m = 0$, particles near the surface move towards the center-line and particles at depth move away form the center-line. For large m, all particles have negligible horizontal movements. Low rotational modulus allows more freedom of particle movements. For example, the settlements at center-line under the pressure load are 0.0049 in and 0.0034 in respectively for the $m = 0$ and 8 cases.

Although the vertical stress distribution shows agreement in patterns for three cases, the magnitudes are somewhat different. This can be seen from the computed vertical stress along depth plotted in Fig. 8. The rate of decrease is much faster for the high rotation modulus material. The vertical stress along the horizontal direction at depth B, as given in Fig. 8, spreads into a wider area than that of the $m = 0$ case.

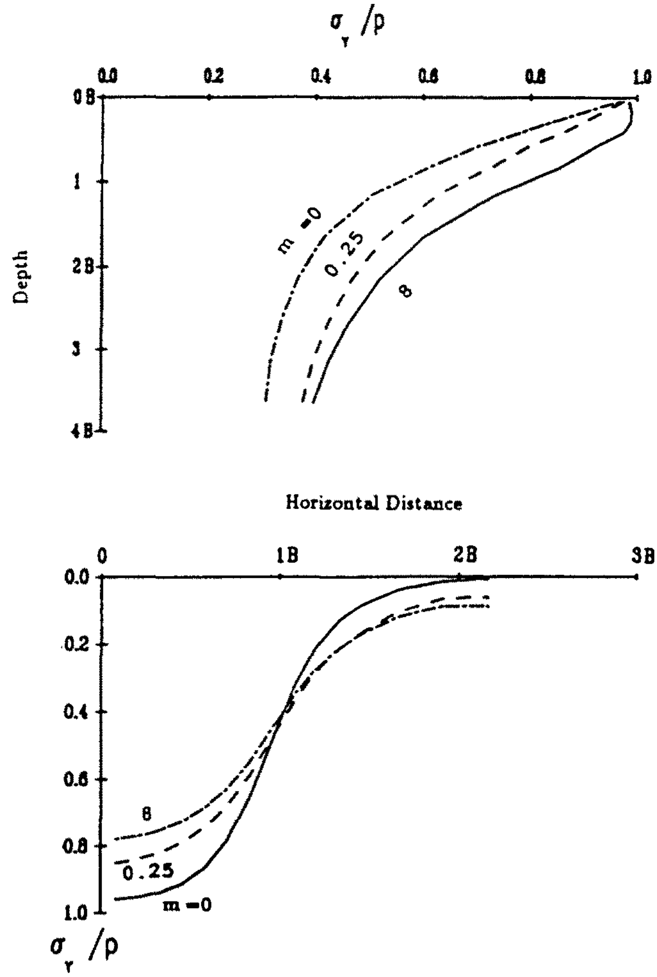

Fig. 8. Vertical stress distribution computed for various rotational modulus.

The effect of m is also substantial on shear stress as shown in Fig. 9. For the case of $m = 0$, the stress $\sigma_{xy} = \sigma_{yx} = 40$ psi. However, when there is a rotational moduli, the value of σ_{xy} is not equal to σ_{yx} due to the presence of couple stress. For the case of $m = 8$, $\sigma_{xy} = 97$ psi while $\sigma_{yx} = 5$ psi. Due to the asymmetry of shear stress, particles spin. This effect will produce plastic shear sliding in certain preferred orientations if the material is not elastic in nature.

The computed distribution of couple stress μ_{xz} is plotted in Fig. 10. The magnitude of couple stress increases with the modulus M_r. Note that large couple stresses are observed in the region adjacent to the pressure loading. It is consistent with the presence of a rotation gradient in this region as previously shown in Fig. 6.

Based on the results of this example, particle rotation and couple stress have considerable effects on the over-all deformation behavior of the assembly.

6. SUMMARY AND CONCLUSION

The granular material perceived as a collection of particles is modelled as a macro-continuum taking account of the structural micro-discreteness of the material and the particle interaction. The constitutive constants are derived in closed-form for granular solid with isotropic fabric distribution and linear-elastic contact properties. This expression is useful in providing a clearer understanding of the physical meaning of the constitutive constants in relation to stress–strain behavior.

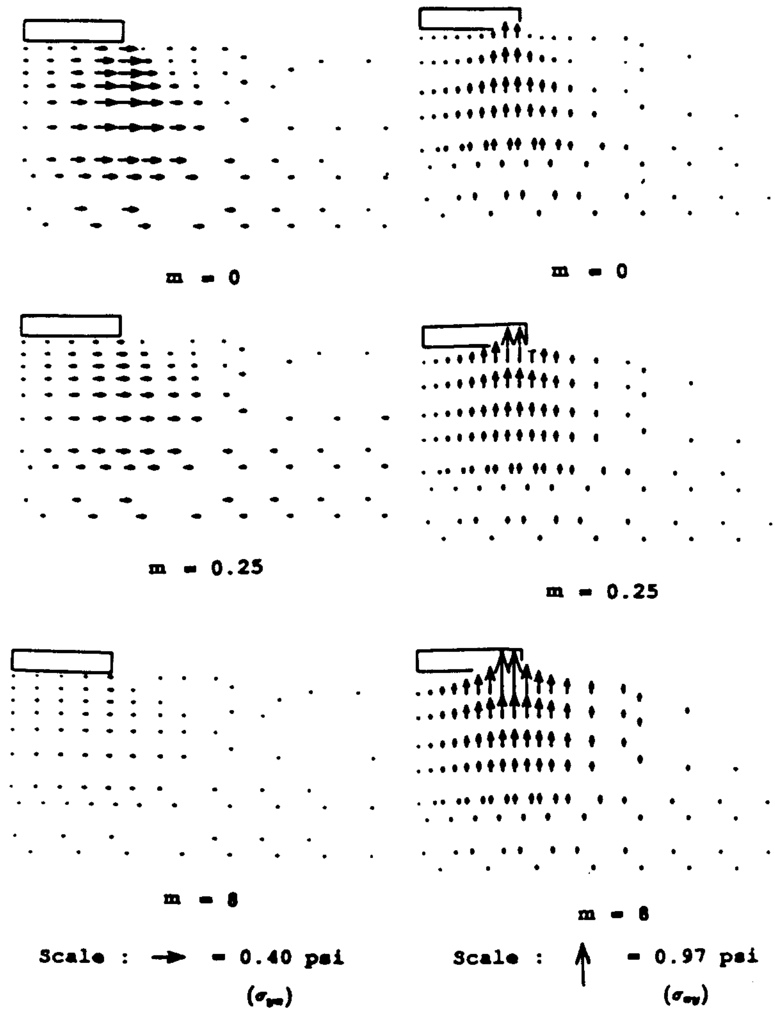

Fig. 9. Computed fields of shear stresses.

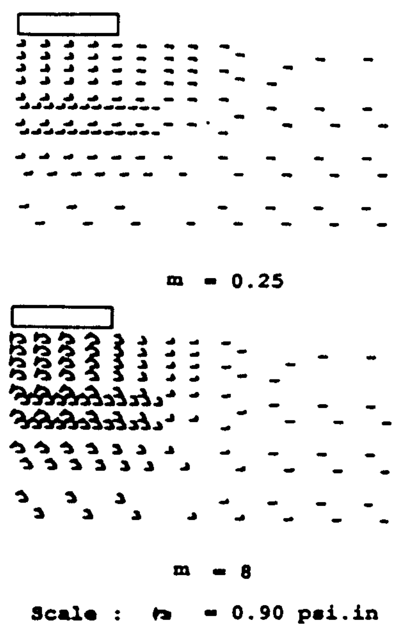

Fig. 10. Computed fields of couple stress.

According to the derivation, six constants are identified: three constants λ, G, z relating stretch, stress and strain, and three constants λ_r, G_r, z_r relating couple stress and polar strain. Of the six constants, four are independent corresponding to the four inter-particle stiffness k_n, k_t, g_n and g_t.

Material constants λ and G are the usual Lamé constants. The material constant z, termed as spin modulus, relates particle spin and asymmetrical shear stress. The spin modulus imposes constraints of particle rotations of the system which is not present in classic elasticity. Therefore, even for the condition of symmetric shear stress (i.e. without the presence of couple stress), the stress–strain behavior differs from that of the conventional elasticity.

Parallel to the constants λ, G, z, the three material constants λ_r, G_r, z_r are needed when couple stress is considered. These three constants govern the spatial variation of particle rotation which is caused by the contact-couples transmitting in the granular media through the rolling or twisting stiffness at contacts between particles.

A finite element analysis incorporating these constitutive constants is used to analyze examples for granular material under boundary pressure. Under the condition of zero rotational contact stiffness (i.e. $M_r = 0$; that is without couple stress), the present model computes settlement and stress distribution which are fairly different from that obtained from elasticity. The difference is attributed to the effect of spin modulus. As a result, the computed settlement is higher and the vertical stress transmits to deeper depth.

When rotational stiffness of inter-particle contact is considered, the granular media is able to transmit couple stress. The effects of M_r on the deformation behavior are substantial. The computed settlement decreases as M_r increases.

Since spin and rotational modulus for granular material depend on the properties of inter-particle contact, it is rational to account their effects in the anlysis of the over-all deformation behavior of granular material.

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